Lecture 7 - Sep 25

Math Review

Relational Overriding
Functional Property
Partial Functions vs. Total Functions

Announcements/Reminders

- Today's class: notes template posted
- Event-B Summary Document
- Priorities:
  + Lab1 → Review
  + Lab2 → Review
- Released:
  + ProgTest guide
  + 2 Practice Tests and solutions
  + Lab1, Lab2 solutions
  + Possible change of ProgTest venue – to be confirmed

Relational Operations: Overriding

$\{x \mid P(x) \vee Q(x)\}$
$= \{x \mid P(x)\} \cup \{x \mid Q(x)\}$

$r = \{\cancel{(a, 1)}, (b, 2), \cancel{(c, 3)}, \cancel{(a, 4)}, (b, 5), \cancel{(c, 6)}, (d, 1), (e, 2), (f, 3)\}$

Example: Calculate r overridden with $\{(a, 3), (c, 4)\}$

Hint: Decompose results to those in t's domain and those not in t's domain.

$$\boxed{\underset{\text{relation}}{r} \mathbin{\lhd\!\!-} \underset{\text{relation}}{t}} = \{(d, r') \mid (d, r') \in t \vee ((d, r') \in r \wedge d \notin dom(t))\}$$

r overridden by t

$$= \{(d, r') \mid (d, r') \in t\} \cup \{(d, r') \mid (d, r') \in r \wedge d \notin dom(t)\}$$

$$= t \cup (\underbrace{\boxed{dom(t)}}_{\{a, c\}} \mathbin{\lhd\!\!\!-} r)$$

$$= \{(a, 3), (c, 4)\} \cup \{(b, 2), (b, 5), (d, 1), (e, 2), (f, 3)\}$$

$r = \{\cancel{(a, 1)}, (b, 2), \cancel{(c, 3)}, \cancel{(a, 4)}, (b, 5), \cancel{(c, 6)}, (d, 1), (e, 2), (f, 3)\}$

(a, 3), (c, 4)

Example: Calculate r overridden with $\{(a, 3), (c, 4)\}$

t

$r \in$ Alphabet $\leftrightarrow \mathbb{Z}$

Lab 1 b : Account $\nrightarrow \mathbb{Z}$

transfer from acc1 to acc2

original rel'l before transfer.

proposed changes

After Transfer

$b := b \triangleleft \{acc1 \mapsto \ldots, acc2 \mapsto \ldots\}$

$r \triangleleft t$

changed version of r according to t

basically r, except all pairs with first elements in dom(t), they must agree with t.

Exercises: Algebraic Properties of Relational Operations

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

Define the image of set s on r in terms of other relational operations.

Hint: What range of value should be included?

$\boxed{r[s]}$ ✗ $= s \lhd r$

$\subseteq ran(r)$

$\boxed{① \; s \lhd r}$

② $s \mathbin{⩤} r$

③ $r \rhd s$

④ $r \mathbin{⩥} s$

$$r[s] = ran(s \lhd r)$$

Define r overridden with set t in terms of other relational operations.

Hint: To be in t's domain or not to be in t's domain?

$$r \oplus t = t \cup \{dom(t) \mathbin{⩤} r\}$$

Functional Property

$r \in S \leftrightarrow T$

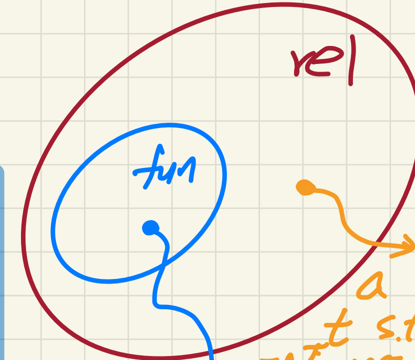
$$\text{isFunctional}(r) \Leftrightarrow$$
$$\forall s, t1, t2 \bullet$$
$$(\boxed{s} \in S \land \boxed{t1} \in T \land \boxed{t2} \in T)$$
$$\Rightarrow$$
$$(\boxed{(s, t1) \in r \land (s, t2) \in r} \Rightarrow \boxed{t1 = t2})$$

range values.

a domain value

two pairs sharing the same 1st elements are $\in r$

2nd elements in two pairs must be the same.



a rel t s.t. isFunction(t) is false

a rel. r s.t. isFunction(r) is true.

Q: Smallest relation satisfying the functional property.

$\emptyset \rightarrow$ can't find witness of violating func. property

You cannot have the same domain value s mapping to two distinct range values $t1$ and $t2$

e.g. $r = \{\boxed{(a, 1)}, (b, 2), \boxed{(a, 3)}\}$

$T \quad \boxed{(a, 1) \in r \land (a, 3) \in r}$

witness of violation $\leftarrow \Rightarrow \boxed{1 = 3} \quad F$

Functional Property

* Each domain value maps to at most one range value

isFunctional(r) $\Leftrightarrow$

$$\forall s, t1, t2 \bullet$$

$$(s \in S \wedge t1 \in T \wedge t2 \in T)$$

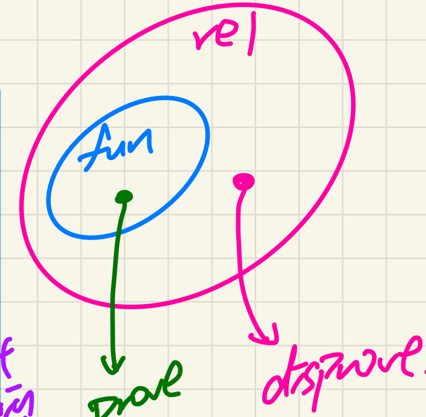
$$\Rightarrow$$

* **

$$(\boxed{(s, t1) \in r \wedge (s, t2) \in r \Rightarrow t1 = t2})$$

** $t1 \neq t2 \Rightarrow \boxed{\neg((s,t_1) \in r \wedge (s,t_2) \in r)}$

two pairs with distinct range values but same domain value cannot exist in the same relation.



Q: How to **prove** or **disprove** that a relation r is a function.

Q: Rewrite the functional property using **contrapositive**.

Prove

① Show that $r = \emptyset$ ($F \Rightarrow \_ \equiv T$)

② Go over all pairs in r, show that each dom. value maps to no more than one values.

Disprove

Find $(s,t_1) \in r$ $(s,t_2) \in r$ but $t1 \neq t2$

$\nrightarrow$ partial

$\longrightarrow$ total

Partial Functions vs. Total Functions

$r \in S \leftrightarrow T$

partial

$r \in S \nrightarrow T \iff (\ \text{isFunctional}(r) \land \boxed{\text{dom}(r) \subseteq S}\ )$

$r \in S \rightarrow T \iff (\ \text{isFunctional}(r) \land \text{dom}(r) = S\ )$

total

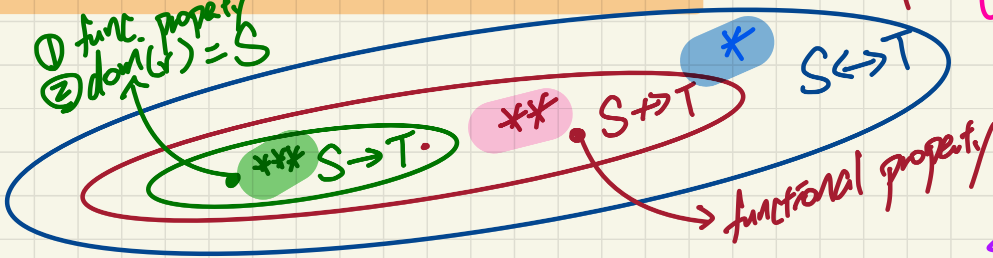
$\boxed{\text{dom}(r) \subset S} \lor \boxed{\text{dom}(r) = S}$

case 1: there's some dom value that does not has corresponding range value

case 2: every dom value has its defined ran value.

Exercise: Visualize $S \nrightarrow T$ vs. $S \longrightarrow T$

Every function is a partial function.



e.g., $\{\ \{(2, a), (1, b)\}, \{(2, a), (3, a), (1, b)\}\ \} \subseteq \boxed{\{1, 2, 3\} \nrightarrow \{a, b\}}$

set of all possible partial functions between S and T

S T

only check func. property.

e.g., *** $\{(2, a), (3, a), (1, b)\} \in \boxed{\{1, 2, 3\} \rightarrow \{a, b\}}$

$\text{dom}(r) = \{1, 2, 3\} = S \land$ functional ✓

e.g., ** $\{(2, a), (1, b)\} \notin \{1, 2, 3\} \rightarrow \{a, b\}$

$\text{dom}(r) = \{1, 2\} \subset S$ S

e.g., * $\{(2, a), (1, b), (3, a), (1, a)\} \notin \{1, 2, 3\} \rightarrow \{a, b\}$

violates the fun. property $\Rightarrow$ not a partial funct.

$$\text{e.g., } \{ \overset{r1}{\underline{\{(2, a), (1, b)\}}}, \overset{r2}{\underline{\{(2, a), (3, a), (1, b)\}}} \} \subseteq \underset{S}{\underline{\{1, 2, 3\}}} \nrightarrow \underset{T}{\underline{\{a, b\}}}$$

$$\Leftrightarrow$$

$$r1 \in S \nrightarrow T$$
$$\wedge$$
$$r2 \in S \nrightarrow T$$

$$\text{e.g., } \{ \boxed{\{(2, a), (1, b)\}}, \underline{\{(2, a), (3, a), (1, b)\}} \} \overset{\in}{\cancel{\subseteq}} \boxed{\{1, 2, 3\} \nrightarrow \{a, b\}}$$

a set of sets of ordered pairs

a set of ordered pairs

a set of ordered pairs

a set where each member is a set of ordered pairs (sat. fun property)

$S = \{1, 2, 3\}$

$T = \{a, b\}$

$r = \{(1, a), (2, b), (3, a)\}$

✓ ① r is a relation.

✓ ② r is a partial function.

(most acc). ✓ ✓ ③ r is a total function.

$f(n) = 2n^2 + 3n - 4$

most accurate! ✓ (1) $f(n)$ is $O(n^2)$

✓ (2) $f(n)$ is $O(n^3)$

✓ (3) $f(n)$ is $O(2^n)$

Q1. Correct.

Q2. Most accurate?